Week 12

Recall:

The gradient flow of ϕ, $x:[0,+\infty)\to H$, starting at x_0 is:

$$(GF)\quad \begin{cases} \dot{x}(t) = -\nabla\phi(x(t)) \\ x(0) = x_0 \end{cases}$$

$\to$ if $\phi \notin C^1$ but convex, we can replace by $\dot{x} \in -\partial\phi(x(t))$.

The implicit Euler scheme gives a way to approximate this problem. Given a fixed time-step τ, we define a "discrete solution" $x^\tau(t) = x_k^\tau$ if $t\in[(k-1)\tau, k\tau)$,

where $\begin{cases} x_0^\tau = x_0 \\ x_{k+1}^\tau \text{ solves } \dfrac{x-x_k^\tau}{2\tau} \in -\partial\phi(x). \end{cases}$ $\left(\text{or equivalently minimizes } \dfrac{\|x-x_k^\tau\|^2}{2\tau} + \phi(x)\right)$

$\to d(x,x_k^\tau)^2$ this makes sense in metric spaces

The Dirichlet energy functional

Let $H = L^2(\mathbb{R}^d)$, $\phi(u) = \begin{cases} \frac{1}{2}\int|\nabla u|^2 & \text{if } u\in W^{1,2}(\mathbb{R}^d) \\ +\infty & \text{otherwise.}\end{cases}$

Prop: The GF of ϕ wrt L^2 scalar product is the heat equation,

$$\underbrace{\partial_t u(t)}_{\in L^2(\mathbb{R}^d)} \in -\partial\phi(u(t)) \iff \partial_t u(t,x) = \Delta u(t,x) \text{ distributionally.}$$

(that is: $0 = \int(\partial_t\varphi(t,x)u(t,x) + \Delta\varphi(t,x)u(t,x))$ $\forall\varphi\in C_c^\infty$.)

Proof: Claim: $\partial\phi(u) = \begin{cases} -\Delta u & \text{if } \Delta u\in L^2 \\ \emptyset & \text{otherwise.}\end{cases}$

$p\in\partial\phi(u) \iff \forall x = u+w,\ \phi(u+w)-\phi(u) \geq \langle p,w\rangle_{L^2}$,

$\forall u,w:\quad \phi(u+w)-\phi(u) = \frac{1}{2}\int\left(|\nabla u+\nabla w|^2 - |\nabla u|^2\right) = \int\langle\nabla u,\nabla w\rangle + \frac{|\nabla w|^2}{2} \quad (*)$

- If $\Delta u\in L^2$, RHS of $(*)$ $= -\int\Delta u\, w + \overbrace{\int\frac{|\nabla w|^2}{2}}^{\geq 0} \geq \langle -\Delta u, w\rangle_{L^2} \Rightarrow -\Delta u\in\partial\phi(u)$.

- Let now $p\in\partial\phi(u)$. Then $\Delta u\in L^2$ and $p=-\Delta u$: indeed, $\forall w$,

$\int\langle\nabla u,\nabla w\rangle + \frac{|\nabla w|^2}{2} \geq \int pw$. Now fix w_0 and take $w=\varepsilon w_0$:

$$\cancel{\varepsilon}\int\langle\nabla u,\nabla w_0\rangle + \frac{\varepsilon^{\cancel{2}}}{2}\int|\nabla w_0|^2 \geq \cancel{\varepsilon}\int pw_0.$$ Exchanging w_0 with $-w_0$

we get equality, so $\forall w_0\in C_c^\infty$,

$$\underbrace{\int\langle\nabla u,\nabla w_0\rangle}_{\int(-\Delta u)w_0} = \int p w_0, \text{ then } -\Delta u = p\in L^2(\mathbb{R}^d). \qquad \square$$

$\to$ as a distribution

Remark: Let ϕ be a convex, C^1 function. Then GF is unique:

Indeed, take $x(t), y(t)$ GF from x_0, y_0:

$$\frac{1}{2}\frac{d}{dt}\|x(t)-y(t)\|^2 = \langle x(t)-y(t), \dot{x}(t)-\dot{y}(t)\rangle = (*) =$$

ϕ convex

$$= \langle x(t)-y(t), -\nabla\phi(x(t))+\nabla\phi(y(t))\rangle \leq 0$$

Applied with $x_0 = y_0$ we get uniqueness. $\partial\phi(y(t))$ $\ni$ — subdifferential of convex is monotone.

If ϕ not C^1, $(*) = \langle x(t)-y(t), -p(t)+q(t)\rangle \leq 0$, $p(t) \in \partial\phi(x(t))$.

The heat equation as a gradient flow (#2)

A natural way to approximate the heat equation is

$u^\tau(\tau k) = u_k^\tau$ is the min in L^2 of $\dfrac{\|u - u^\tau(\tau(k-1))\|_{L^2}^2}{2\tau} + \int|\nabla u|^2$

u^τ is then extended piecewise constant.

.... The heat equation has another GF structure!

Define given $\tau > 0$,

bounded convex domain in $\mathbb{R}^d$

$$\rho_k^\tau dx \text{ is the min in } \mathcal{P}(\Omega) \text{ of } \rho \mapsto \frac{W_2^2(\rho, \rho_{k-1}^\tau)}{2\tau} + \int \rho\log\rho$$

$\hookrightarrow$ slight abuse of notation, denoting ρ_k^τ both the measure and its density wrt Lebesgue.

Remark:
- We can assume $\rho_0 \in \mathcal{P}(\Omega)$ by rescaling.
- We assume $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ with $\int \rho_0\log\rho_0 < +\infty$

At τ fixed, we associate to the seq. $\rho^\tau(t,x) = \rho_k^\tau$ if $t \in ((k-1)\tau, k\tau)$

$\to$ next week

Theorem (Jordan, Kinderlehrer, Otto; 1998):

There exists $\rho \in L^1_{loc}([0,+\infty)\times\Omega)$ such that $\rho^\tau \to \rho$ up to a subsequence and ρ is a solution to the heat equation starting from ρ_0.

Lemma: Given $\bar\rho \in \mathcal{P}(\Omega)$ (open) s.t. $\int \bar\rho\log\bar\rho < +\infty$, then

$\exists\, \rho_\infty \in \mathcal{P}(\Omega)$, $\rho_\infty \ll dx$, min of $\rho \mapsto \dfrac{W_2^2(\rho,\bar\rho)}{2\tau} + \int\rho\log\rho$

Proof: Let $\{\rho_m\}_m \subseteq \mathcal{P}(\Omega)$ be a minimizing sequence:

$$\frac{W_2^2(\rho_m,\bar\rho)}{2\tau} + \int \rho_m \log\rho_m \longrightarrow \inf_\rho \frac{W_2^2(\rho,\bar\rho)}{2\tau} + \int \rho\log\rho$$

absolut. continuous measure

$\{\rho_m\}_m \subseteq \mathcal{P}(\bar\Omega)$ $\underbrace{\text{are tight}}_{\text{defined on compact set}}$ up to a subsequence, by Prokhorov: $\rho_m \xrightarrow{*} \rho_\infty$ measure, with $\int_{\bar\Omega}\rho_\infty = 1$

Claim 1: $\int_{\underbrace{\Omega}_{\neq\bar\Omega}} \rho_\infty = 1$ **Claim 2**: $\rho_\infty \ll dx$ (later)

This is nontrivial: a sequence bounded in $L^2(L^1) \Longrightarrow$ any weak limit is L^2 (measure)

Conclusion of the proof (with the two claims): $S \mapsto S\log S$ convex implies

$$\int_\Omega \rho_\infty \log\rho_\infty \le \liminf_m \int_\Omega \rho_m \log\rho_m \quad \boxed{(1)}$$

$\hookrightarrow$ analogous to $\|\rho_\infty\|_{L^p} \le \liminf_m \|\rho_m\|_{L^p}$, suggestion in lecture notes.

Look at $W_2^2(\rho_m,\bar\rho)$, $\rho_m \longrightarrow \rho_\infty$:

$\shortparallel$ $\quad$ optimal between ρ_m and $\bar\rho \Rightarrow \gamma_m$ tight $\qquad$ stability theory

$$\int \underbrace{|x-y|^2}_{\text{continuous \& bdd in } \Omega\times\Omega} d\gamma_m \Longrightarrow \gamma_{m_k} \longrightarrow \gamma \quad (\gamma \text{ transport plan between } \rho_\infty \text{ and } \bar\rho).$$

in fact, equality

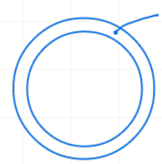
$$\hookrightarrow \int |x-y|^2 d\gamma \ge W_2^2(\rho_\infty,\bar\rho) \quad \boxed{(2)}$$

From $\boxed{(1)}$ and $\boxed{(2)}$:

$$\int \rho_\infty\log\rho_\infty + \frac{W_2^2(\rho_\infty,\bar\rho)}{2\tau} \le \liminf_m \int \rho_m\log\rho_m + \frac{W_2^2(\rho_m,\bar\rho)}{2\tau}$$

by choice of ρ_m

Then ρ_∞ is a minimizer. $\qquad = \inf \int \rho\log\rho + \dfrac{W_2^2(\rho,\bar\rho)}{2\tau}$



Proof of claim 1: Let $N_\varepsilon = \{x : \operatorname{dist}(x,\Omega^c) < \varepsilon\}$

For ε small, $|N_\varepsilon| < C\varepsilon$,

because $\int\rho_m\log\rho_m \le \inf + 1$

$$\int_{N_\varepsilon}\rho_m \le \int_{N_\varepsilon\cap\{\rho_m<L\}}\rho_m + \int_{N_\varepsilon\cap\{\rho_m\ge L\}}\rho_m\frac{\log\rho_m}{\log L} \le LC\varepsilon + \frac{C}{\log L} \le (*).$$

Choose L s.t. RHS small in ε, $L = \varepsilon^{-1/2}$,

$$(*) \le \varepsilon^{1/2} + \frac{2C}{\log\varepsilon^{-1}} \le \frac{C}{|\log\varepsilon|} \longrightarrow 0 \quad \text{as } \varepsilon\to 0 \text{ unif. in } m, \text{ so:}$$

cut-off $= \begin{cases} 1 & \Omega\setminus N_\varepsilon \\ 0 & N_{\varepsilon/2}\end{cases}$

$$\underset{\substack{\downarrow \\ \int\rho_\infty\varphi_\varepsilon}}{\int\rho_m\varphi_\varepsilon} \ge \int_{\Omega\setminus N_\varepsilon}\rho_m \ge 1 - \frac{C}{|\log\varepsilon|}. \qquad \varepsilon \text{ arbitrary} \Longrightarrow \int_\Omega \rho_\infty = 1.$$

Proof of claim 2: Consider, given $M>0$, $\{\rho_m \wedge M\}$ bounded in L^∞.

By Banach-Alaoglu, $\rho_{m_\ell} \wedge M \xrightarrow{*} \rho_M$ in L^∞

$L^1 \downarrow$ $\longrightarrow$ ρ_M increasing sequence in M, converges pointwise to $\overline{\rho_\infty}(x) := \sup_M \rho_M(x)$

$\overline{\rho_\infty}$

$$\int_\Omega (\rho_m - \rho_m \wedge M) = \int_{\{\rho_m > M\}} (\rho_m - M)\,dx \leq \frac{1}{\log M} \int_{\{\rho_m > M\}} \overbrace{(\rho_m \log \rho_m + 1)}^{\geq 0}\,dx$$

$\overline{\rho_\infty}$ is in L^1, by monotone convergence theorem

$$\leq \frac{1}{\log M} \int_\Omega (\rho_m \log \rho_m + 1)\,dx \leq \frac{C}{\log M}.$$

"ρ_m and $\rho_m \wedge M$ are close in L^1 uniformly in m".

Then, $\exists$ subseq. $\rho_{m_{\ell_M}} \wedge M \xrightarrow[M\to\infty]{} \overline{\rho_\infty}$ in L^1

this is close to $\rho_{m_{\ell_M}}$ up to $\dfrac{C}{\log M}$

$\downarrow$

$= \rho_\infty$